CHAPTER 3

Quantum distribution theory and partially coherent radiation

As we have seen in the previous chapters, there are quantum fluctuations associated with the states corresponding to classically well-defined electromagnetic fields. The general description of fluctuation phenomena requires the density operator. However, it is possible to give an alternative but equivalent description in terms of distribution functions. In the present chapter, we extend our treatment of quantum statistical phenomena by developing the theory of quasi-classical distributions. This is of interest for several reasons.

First of all, the extension of the quantum theory of radiation to involve nonquantum stochastic effects such as thermal fluctuations is needed. This is an important ingredient in the theory of partial coherence. Furthermore, the interface between classical and quantum physics is elucidated by the use of such distributions. The arch type example being the Wigner distribution.*

In this chapter, we introduce various distribution functions. These include the coherent state representation or the Glauber–Sudarshan P-representation. The P-representation is used to evaluate the normally ordered correlation functions of the field operators. As we shall see in the next chapter, the P-representation forms a correspondence between the quantum and the classical coherence theory. This distribution function does not have all of the properties of the classical distribution functions for certain states of the field, e.g., it can be negative. We also discuss the so-called Q-representation associated with the antinormally

* The first quasiclassical distribution, Wigner [1932], was written from a wave function perspective. The later work of Moyal [1949] introduced the characteristic function approach to obtaining the Wigner distribution. For reviews of the subject see Hillery, O'Connell, Scully, and Wigner [1984], and Reichl, chapter 7 [1980]. The very readable textbooks by Louisell [1974], Walls and Milburn [1994], and Cohen [1995] extend the quasiclassical distribution concept and are recommended reading.

ordered correlation functions. Other distribution functions and their properties are also presented.

3.1 Coherent state representation

The study of the interface between quantum and classical physics is a fascinating subject. Nowhere is this better illustrated than in quantum optics, where we are often faced with the problem of characterizing fields which are nearly classical but have important quantum features. The coherent states are well suited to such studies. In order to see why this is the case, let us recall that for a fluctuating *classical* field we are generally dealing with a probability distribution $P(\mathscr{E})$ for the complex field amplitude $\mathscr{E} = |\mathscr{E}|e^{i\phi}$ as indicated in Fig. 3.1.

Now in quantum mechanical problems, a probability distribution for the system comes from the statistical or density operator which is defined as follows. Suppose we know that the system is in state $|\psi\rangle$, then an operator O has the expectation value

$$\langle O\rangle_{\text{QM}} = \langle\psi|O|\psi\rangle, \tag{3.1.1}$$

but we typically do not know that we are in $|\psi\rangle$. We only have a probability P_ψ for being in this state so we must perform an ensemble average as well

$$\langle\langle O\rangle_{\text{QM}}\rangle_{\text{ensemble}} = \sum_\psi P_\psi\langle\psi|O|\psi\rangle. \tag{3.1.2}$$

Now using completeness $\sum_n |n\rangle\langle n| = 1$

$$\begin{aligned}\langle\langle O\rangle\rangle &= \sum_n\sum_\psi P_\psi\langle\psi|O|n\rangle\langle n|\psi\rangle \\ &= \sum_n\sum_\psi P_\psi\langle n|\psi\rangle\langle\psi|O|n\rangle \\ &= \sum_n\langle n|\rho O|n\rangle. \end{aligned}\tag{3.1.3}$$

Thus the radiation field is, in general, described by the density operator

$$\rho = \sum_\psi P_\psi|\psi\rangle\langle\psi|, \tag{3.1.4}$$

where P_ψ is the probability of being in the state $|\psi\rangle$. The expectation value of any field operator O is then given by

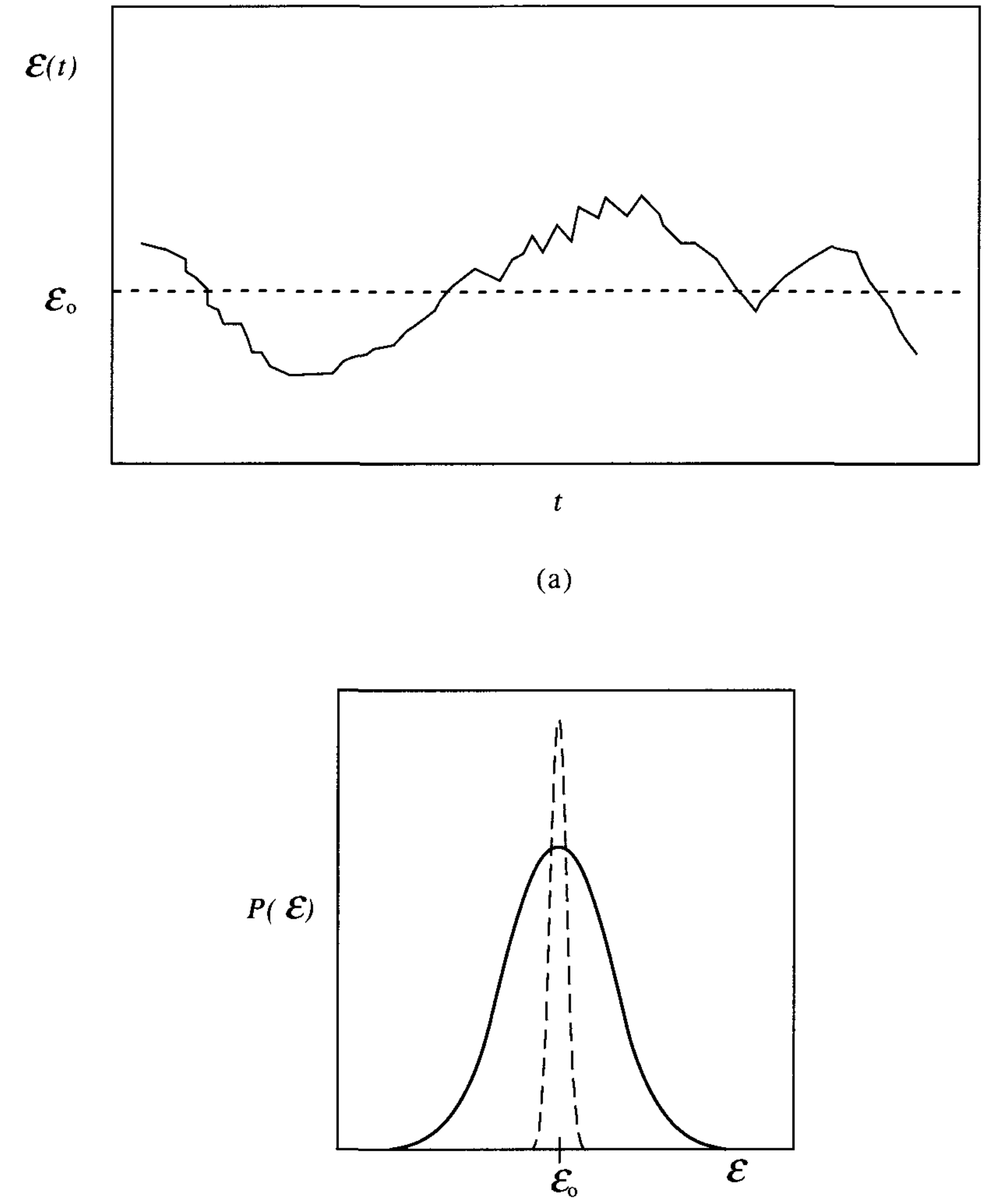


Fig. 3.1
(a) The fluctuating classical field as a function of time for a field with large fluctuations (solid line) and a well stabilized field (dashed line), and (b) associated probability distributions.

$$\langle O \rangle = \mathrm{Tr}(O\rho), \tag{3.1.5}$$

where Tr stands for trace. Now the density operator ρ can be expanded in terms of the photon occupation number states:

$$\rho = \sum_n \sum_m |n\rangle\langle n|\rho|m\rangle\langle m| = \sum_n \sum_m \rho_{nm}|n\rangle\langle m|. \tag{3.1.6}$$

Likewise the expansion may be made in terms of coherent states as

$$\rho = \int \int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}|\alpha\rangle\langle\alpha|\rho|\beta\rangle\langle\beta|. \tag{3.1.7}$$

Following Glauber's convention we define the R-representation as

$$R(\alpha^*, \beta) = \langle\alpha|\rho|\beta\rangle e^{\frac{1}{2}(|\alpha|^2+|\beta|^2)}, \tag{3.1.8}$$

so that the density matrix may be written as

$$\rho = \int\int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}|\alpha\rangle\langle\beta|\, R(\alpha^*, \beta)e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}. \tag{3.1.9}$$

We thus have used two indices n and m or α and β in order to specify the density matrix.

We next make contact with $P(\mathscr{E})$, as discussed earlier, by developing a *diagonal* coherent state representation. That is, we express the density operator ρ in terms of the diagonal pair $|\alpha\rangle\langle\alpha|$ in the following.

3.1.1 Definition of the coherent state representation

Consider an operator $O_N(a, a^\dagger)$, which is a function of a and $a^\dagger$ in the normal order (all the creation operators $a^\dagger$ on the left-hand side and all the annihilation operators a on the right-hand side), i.e.,

$$O_N(a, a^\dagger) = \sum_n\sum_m c_{nm}(a^\dagger)^n a^m. \tag{3.1.10}$$

It may be noted that any operator involving a and $a^\dagger$ can be converted into a normal ordered form by using the commutation relation $[a, a^\dagger] = 1$. For example $a^2a^\dagger = a^\dagger a^2 + 2a$. The expectation value of the operator $O_N(a, a^\dagger)$ can then be written as

$$\begin{aligned}\langle O_N(a, a^\dagger)\rangle &= \mathrm{Tr}[\rho O_N(a, a^\dagger)] \\ &= \sum_n\sum_m c_{nm}\mathrm{Tr}[\rho(a^\dagger)^n a^m].\end{aligned} \tag{3.1.11}$$

As discussed in Appendix 3.A, we define the operator

$$\begin{aligned}&\delta(\alpha^* - a^\dagger)\delta(\alpha - a) \\ &= \frac{1}{\pi^2}\int \exp[-\beta(\alpha^* - a^\dagger)]\exp[\beta^*(\alpha - a)]d^2\beta,\end{aligned} \tag{3.1.12a}$$

or, in an equivalent form

$$\begin{aligned}&\delta(\alpha^* - a^\dagger)\delta(\alpha - a) \\ &= \frac{1}{\pi^2}\int \exp[-i\beta(\alpha^* - a^\dagger)]\exp[-i\beta^*(\alpha - a)]d^2\beta.\end{aligned} \tag{3.1.12b}$$

We will use (3.1.12a) and (3.1.12b) interchangeably in the text. Equation (3.1.11) can then be rewritten as

$$\begin{aligned}\langle O_N(a, a^\dagger)\rangle &= \int d^2\alpha\sum_n\sum_m c_{nm}\mathrm{Tr}[\rho\delta(\alpha^* - a^\dagger)\delta(\alpha - a)](\alpha^*)^n\alpha^m \\ &= \int d^2\alpha P(\alpha, \alpha^*)O_N(\alpha, \alpha^*),\end{aligned} \tag{3.1.13}$$

where

$$P(\alpha, \alpha^*) = \mathrm{Tr}[\rho\delta(\alpha^* - a^\dagger)\delta(\alpha - a)]. \tag{3.1.14}$$

It is seen from Eq. (3.1.13) that the function $P(\alpha, \alpha^*)$ can be used to evaluate the expectation values of any normal ordered function of a and $a^\dagger$ using the methods of classical statistical mechanics. Due to the Hermiticity of the density operator ρ, the distribution function $P(\alpha, \alpha^*)$ is real. Moreover, since $\mathrm{Tr}(\rho) = 1$, $P(\alpha, \alpha^*)$ is normalized to unity, i.e.,

$$\int P(\alpha, \alpha^*)d^2\alpha = 1. \tag{3.1.15}$$

The function $P(\alpha, \alpha^*)$ is referred to as the P-representation or the coherent state representation. The name *coherent state representation* is due to the following representation of the density operator ρ by means of a diagonal representation in terms of the coherent states:

$$\rho = \int P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha|d^2\alpha. \tag{3.1.16}$$

The equivalence of the definitions of $P(\alpha, \alpha^*)$ as given by Eqs. (3.1.14) and (3.1.16) can be seen simply by substituting for ρ from Eq. (3.1.16) into Eq. (3.1.14). As we shall see in the next chapter, $P(\alpha, \alpha^*)$ forms a connection between the classical and quantum coherence theory.

Before considering some examples of the P-representation, we give a simple procedure to find $P(\alpha, \alpha^*)$ from a knowledge of ρ. Let $|\beta\rangle$ and $|-\beta\rangle$ be the coherent states with β and $-\beta$ being the eigenvalues of a, respectively. Then, using Eq. (2.4.7),

$$\begin{aligned}\langle -\beta|\rho|\beta\rangle &= \int P(\alpha, \alpha^*)\langle -\beta|\alpha\rangle\langle\alpha|\beta\rangle d^2\alpha \\ &= e^{-|\beta|^2}\int [P(\alpha, \alpha^*)e^{-|\alpha|^2}]e^{\beta\alpha^* - \beta^*\alpha}d^2\alpha. \end{aligned}\tag{3.1.17}$$

At this point we note that if $\alpha = x_\alpha + iy_\alpha$ and $\beta = x_\beta + iy_\beta$, then $d^2\alpha = dx_\alpha dy_\alpha$ and $\beta\alpha^* - \beta^*\alpha = 2i(y_\beta x_\alpha - x_\beta y_\alpha)$, and Eq. (3.1.17) becomes

$$\begin{aligned}&\langle -\beta|\rho|\beta\rangle e^{|\beta|^2} \\ &= \int\int [P(x_\alpha, y_\alpha)e^{-(x_\alpha^2 + y_\alpha^2)}]e^{2i(y_\beta x_\alpha - x_\beta y_\alpha)}dx_\alpha dy_\alpha. \end{aligned}\tag{3.1.18}$$

Thus, $\langle -\beta|\rho|\beta\rangle e^{|\beta|^2}$ is the two-dimensional Fourier transform of $P(\alpha, \alpha^*)e^{-|\alpha|^2}$. This shows the utility of considering the matrix element $\langle -\beta|\rho|\beta\rangle$, since the inverse Fourier transform readily gives $P(\alpha, \alpha^*)$ in terms of the density operator ρ. On taking the Fourier inverse of Eq. (3.1.17), we obtain

$$\begin{aligned} P(\alpha,\alpha^*) &= \frac{e^{(x_\alpha^2+y_\alpha^2)}}{\pi^2}\int\int\langle-\beta|\rho|\beta\rangle e^{(x_\beta^2+y_\beta^2)}e^{2i(y_\alpha x_\beta - x_\alpha y_\beta)}dx_\beta dy_\beta \\ &= \frac{e^{|\alpha|^2}}{\pi^2}\int\langle-\beta|\rho|\beta\rangle e^{|\beta|^2}e^{-\beta\alpha^*+\beta^*\alpha}d^2\beta. \end{aligned} \tag{3.1.19}$$

This is the required expression.

3.1.2 Examples of the coherent state representation

As a first example, we calculate $P(\alpha,\alpha^*)$ for the thermal field. A field emitted by a source in thermal equilibrium at temperature T is described by a canonical ensemble

$$\rho = \frac{\exp(-\mathscr{H}/k_\mathrm{B}T)}{\mathrm{Tr}[\exp(-\mathscr{H}/k_\mathrm{B}T)]}, \tag{3.1.20}$$

where k_B is the Boltzmann constant and $\mathscr{H}$ is the free-field Hamiltonian, $\mathscr{H} = \hbar\nu(a^\dagger a + 1/2)$. For simplicity, we restrict ourselves to a single mode of the field. On substituting this form of $\mathscr{H}$ into Eq. (3.1.20) we obtain

$$\rho = \sum_n\left[1-\exp\left(-\frac{\hbar\nu}{k_\mathrm{B}T}\right)\right]\exp\left(-\frac{n\hbar\nu}{k_\mathrm{B}T}\right)|n\rangle\langle n|. \tag{3.1.21}$$

Correspondingly

$$\langle n\rangle = \mathrm{Tr}(a^\dagger a\rho) = \left[\exp\left(\frac{\hbar\nu}{k_\mathrm{B}T}\right)-1\right]^{-1}. \tag{3.1.22}$$

Equation (3.1.21) can therefore be rewritten in terms of $\langle n\rangle$ as

$$\rho = \sum_n \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}|n\rangle\langle n|. \tag{3.1.23}$$

This leads to the well-known result that the photon distribution in a thermal field is described by the Bose–Einstein distribution, i.e.,

$$\begin{aligned} \rho_{nn} &= \langle n|\rho|n\rangle \\ &= \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}. \end{aligned} \tag{3.1.24}$$

Next we substitute for ρ from Eq. (3.1.23) into Eq. (3.1.19). We note that

$$\begin{aligned}\langle -\beta|\rho|\beta\rangle &= \sum_n \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}\langle -\beta|n\rangle\langle n|\beta\rangle \\ &= \frac{e^{-|\beta|^2}}{1+\langle n\rangle}\sum_{n=0}^{\infty}\frac{(-|\beta|^2)^n}{n!}\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^n \\ &= \frac{e^{-|\beta|^2}}{1+\langle n\rangle}\exp\left[-|\beta|^2\Big/\left(1+\frac{1}{\langle n\rangle}\right)\right], \qquad (3.1.25)\end{aligned}$$

so that

$$\begin{aligned}P(\alpha,\alpha^*) &= \frac{e^{|\alpha|^2}}{\pi^2(1+\langle n\rangle)}\int e^{-|\beta|^2/\left(1+\frac{1}{\langle n\rangle}\right)}e^{-\beta\alpha^*+\alpha\beta^*}d^2\beta \\ &= \frac{1}{\pi\langle n\rangle}e^{-|\alpha|^2/\langle n\rangle}, \qquad (3.1.26)\end{aligned}$$

i.e., the P-representation of the thermal distribution is given by a Gaussian distribution.

As another example, we consider the P-representation of a coherent state $|\alpha_0\rangle$. Here $\rho = |\alpha_0\rangle\langle\alpha_0|$ so that

$$\begin{aligned}\langle -\beta|\rho|\beta\rangle &= \langle -\beta|\alpha_0\rangle\langle\alpha_0|\beta\rangle \\ &= \exp(-|\alpha_0|^2 - |\beta|^2 - \alpha_0\beta^* + \beta\alpha_0^*). \qquad (3.1.27)\end{aligned}$$

It then follows from Eq. (3.1.19) that

$$\begin{aligned}P(\alpha,\alpha^*) &= \frac{1}{\pi^2}e^{|\alpha|^2-|\alpha_0|^2}\int e^{-\beta(\alpha^*-\alpha_0^*)+\beta^*(\alpha-\alpha_0)}d^2\beta \\ &= \delta^{(2)}(\alpha-\alpha_0), \qquad (3.1.28)\end{aligned}$$

i.e., the P-representation of a coherent state is a two-dimensional delta function.

Even though the P-representation allows us to evaluate the normally ordered correlation functions of the field operators a and $a^\dagger$, it is not nonnegative definite and as such cannot be described as a distribution function for certain field states. This can be readily seen by evaluating the P-representation of a number state $|n\rangle$, for which $\rho = |n\rangle\langle n|$ and

$$\begin{aligned}\langle -\beta|\rho|\beta\rangle &= \langle -\beta|n\rangle\langle n|\beta\rangle \\ &= \exp(-|\beta|^2)\frac{(-1)^n|\beta|^{2n}}{n!}. \qquad (3.1.29)\end{aligned}$$

The corresponding P-representation is, therefore, given by

$$
\begin{aligned}
P(\alpha,\alpha^*) &= \frac{(-1)^n e^{|\alpha|^2}}{\pi^2 n!}\int |\beta|^{2n} e^{-\beta\alpha^*+\beta^*\alpha} d^2\beta \\
&= \frac{e^{|\alpha|^2}}{\pi^2 n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\int e^{-\beta\alpha^*+\beta^*\alpha} d^2\beta \\
&= \frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\delta^{(2)}(\alpha).
\end{aligned}
\tag{3.1.30}
$$

For $n > 0$, this is clearly not a nonnegative definite function and, therefore, a number state does not have a well-defined P-representation.

As we will discuss in the next chapter, whenever the photon distribution ρ_{nn} is narrower than the Poisson distribution, as in the case of number state $|n\rangle$, $P(\alpha,\alpha^*)$ becomes *badly* behaved. This is the price we pay for forcing quantum physics into a classical format, i.e., for using $P(\alpha,\alpha^*)$ instead of say, $R(\alpha,\beta^*)$.

3.2 Q-representation

Just as the P-representation is associated with the evaluation of normally ordered correlation functions of the field operators a and $a^\dagger$, we may define other distribution functions which may be associated with different orderings of a and $a^\dagger$. The distribution function which helps in determining the antinormally ordered correlation functions is the so-called Q-representation. It is defined as

$$
Q(\alpha,\alpha^*) = \mathrm{Tr}[\rho\delta(\alpha-a)\delta(\alpha^*-a^\dagger)]. \tag{3.2.1}
$$

It follows, on inserting the representation (2.4.6) for unity between $\delta(\alpha-a)$ and $\delta(\alpha^*-a^\dagger)$ and using (2.2.1) that

$$
\begin{aligned}
Q(\alpha,\alpha^*) &= \frac{1}{\pi}\mathrm{Tr}\int d^2\alpha'[\rho\delta(\alpha-a)|\alpha'\rangle\langle\alpha'|\delta(\alpha^*-a^\dagger)] \\
&= \frac{1}{\pi}\mathrm{Tr}\int d^2\alpha'\{\rho\delta(\alpha-\alpha')|\alpha'\rangle\langle\alpha'|\delta[\alpha^*-(\alpha')^*]\} \\
&= \frac{1}{\pi}\mathrm{Tr}(\rho|\alpha\rangle\langle\alpha|) \\
&= \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle,
\end{aligned}
\tag{3.2.2}
$$

i.e., $Q(\alpha,\alpha^*)$ is proportional to the diagonal element of the density operator in the coherent state representation. It follows from the completeness of the coherent states $|\alpha\rangle$ (Eq. (2.4.6)) and the condition $\mathrm{Tr}(\rho) = 1$ that $Q(\alpha,\alpha^*)$ is normalized to unity, i.e.,

$$\int Q(\alpha, \alpha^*) d^2\alpha = 1. \tag{3.2.3}$$

In order to see how the antinormally ordered correlation functions of a and $a^\dagger$ are evaluated using the Q-representation, we first define a function $O_A(a, a^\dagger)$ in antinormal order, i.e.,

$$O_A(a, a^\dagger) = \sum_n \sum_m d_{nm} a^n (a^\dagger)^m. \tag{3.2.4}$$

It then follows that

$$\begin{aligned}
\langle O_A(a, a^\dagger)\rangle &= \mathrm{Tr}[O_A(a, a^\dagger)\rho] \\
&= \sum_n \sum_m d_{nm} \mathrm{Tr}[a^n (a^\dagger)^m \rho] \\
&= \sum_n \sum_m d_{nm} \mathrm{Tr}\left[\frac{1}{\pi} \int a^n |\alpha\rangle\langle\alpha| (a^\dagger)^m \rho d^2\alpha\right] \\
&= \sum_n \sum_m d_{nm} \frac{1}{\pi} \int \alpha^n (\alpha^*)^m \langle\alpha|\rho|\alpha\rangle d^2\alpha \\
&= \int Q(\alpha, \alpha^*) O_A(\alpha, \alpha^*) d^2\alpha,
\end{aligned} \tag{3.2.5}$$

where, in the third line, we inserted

$$\frac{1}{\pi} \int |\alpha\rangle\langle\alpha| d^2\alpha = 1. \tag{3.2.6}$$

Unlike the P-representation, $Q(\alpha, \alpha^*)$ is nonnegative definite and bounded. This can be seen by substituting for ρ from Eq. (3.1.4) into Eq. (3.2.2). We then obtain

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} \sum_\psi P_\psi |\langle\psi|\alpha\rangle|^2. \tag{3.2.7}$$

Since $|\langle\psi|\alpha\rangle|^2 \leq 1$, we have

$$Q(\alpha, \alpha^*) \leq \frac{1}{\pi}. \tag{3.2.8}$$

The Q-representation may be related to the P-representation by taking the coherent state diagonal element of ρ in Eq. (3.1.16). The resulting equation is

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} \int P(\alpha', \alpha'^*) e^{-|\alpha - \alpha'|^2} d^2\alpha'. \tag{3.2.9}$$

As an example, $Q(\alpha, \alpha^*)$ for a number state $|n\rangle$ is given by

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} |\langle n|\alpha\rangle|^2 = \frac{e^{-|\alpha|^2} |\alpha|^{2n}}{\pi n!}, \tag{3.2.10}$$

which is a well-behaved function. The Q-representation of a squeezed state is given in Section 3.5.

3.3 The Wigner–Weyl distribution

So far we have discussed various distribution functions, namely P- and Q-representations associated with the normal and the antinormal orderings, respectively, of the operators a and $a^\dagger$. We can similarly derive distribution functions associated with other orderings.

To summarize, we have introduced

$$P(\alpha,\alpha^*) = \mathrm{Tr}[\delta(\alpha^* - a^\dagger)\delta(\alpha - a)\rho], \tag{3.3.1a}$$

$$Q(\alpha,\alpha^*) = \mathrm{Tr}[\delta(\alpha - a)\delta(\alpha^* - a^\dagger)\rho], \tag{3.3.1b}$$

which we can write in terms of the so-called characteristic functions. For example, inserting (3.1.12b) into (3.3.1a) we have

$$P(\alpha,\alpha^*) = \frac{1}{\pi^2}\int d^2\beta e^{-i\beta\alpha^* - i\beta^*\alpha}C^{(n)}(\beta,\beta^*), \tag{3.3.2}$$

where the characteristic function $C^{(n)}(\beta,\beta^*)$ is defined as

$$C^{(n)}(\beta,\beta^*) = \mathrm{Tr}\left(e^{i\beta a^\dagger}e^{i\beta^* a}\rho\right). \tag{3.3.3}$$

Likewise, we may write (3.3.1b) as

$$Q(\alpha,\alpha^*) = \frac{1}{\pi^2}\int d^2\beta e^{-i\beta\alpha^* - i\beta^*\alpha}C^{(a)}(\beta,\beta^*), \tag{3.3.4}$$

with the characteristic function

$$C^{(a)}(\beta,\beta^*) = \mathrm{Tr}\left(e^{i\beta^* a}e^{i\beta a^\dagger}\rho\right). \tag{3.3.5}$$

Another useful distribution, due to Wigner and Weyl, is defined as

$$W(\alpha,\alpha^*) = \frac{1}{\pi^2}\int d^2\beta e^{-i\beta\alpha^* - i\beta^*\alpha}C^{(s)}(\beta,\beta^*), \tag{3.3.6}$$

where the characteristic function $C^{(s)}(\beta,\beta^*)$ is given by

$$C^{(s)}(\beta,\beta^*) = \mathrm{Tr}\left(e^{i\beta a^\dagger + i\beta^* a}\rho\right). \tag{3.3.7}$$

This distribution function $W(\alpha,\alpha^*)$ is associated with symmetric ordering. It can be used to evaluate expectation values of any symmetrically ordered functions of a and $a^\dagger$ in a classical fashion. For example,

$$\frac{1}{2}\langle aa^\dagger + a^\dagger a\rangle = \int W(\alpha,\alpha^*)\alpha\alpha^* d^2\alpha. \tag{3.3.8}$$

In Appendix 3.B, we give a procedure to find the c-number function $O_S(\alpha,\alpha^*)$ corresponding to the symmetrically ordered form of an operator $O(a,a^\dagger)$.

Historically, the $W(\alpha,\alpha^*)$ distribution was introduced in terms of the position $\hat{q}$ and momentum $\hat{p}$ operators in a form equivalent to

$$W(p,q) = \frac{1}{(2\pi)^2} \int d\sigma \int d\tau e^{i(\tau p+\sigma q)} \mathrm{Tr}\left[e^{-i(\tau\hat{p}+\sigma\hat{q})}\rho\right]. \qquad (3.3.9)$$

To cast this into the form first introduced by Wigner we use the operator identity

$$e^{A+B} = e^A e^B e^{-[A,B]/2},$$

which holds when the commutator $[A, B]$ commutes with A and B, to write (3.3.9) as

$$W(p,q) = \frac{1}{(2\pi)^2} \int d\sigma \int d\tau e^{i(\tau p+\sigma q)} \mathrm{Tr}\left(e^{-i\tau\hat{p}} e^{-i\sigma\hat{q}} e^{-i\hbar\sigma\tau/2}\rho\right), \qquad (3.3.10)$$

which by cyclic invariance under the trace may be written as

$$W(p,q) = \frac{1}{(2\pi)^2} \int d\sigma \int d\tau e^{i(\tau p+\sigma q)} \mathrm{Tr}\left(e^{-i\tau\hat{p}/2} e^{-i\sigma\hat{q}} \rho e^{-i\tau\hat{p}/2}\right) e^{-i\hbar\sigma\tau/2}. \qquad (3.3.11)$$

Writing the trace in the coordinate representation this becomes

$$W(p,q) = \frac{1}{(2\pi)^2} \int d\sigma \int d\tau e^{i(\tau p+\sigma q)} \int dq' \langle q'|e^{-i\tau\hat{p}/2} e^{-i\sigma\hat{q}} \rho e^{-i\tau\hat{p}/2}|q'\rangle e^{-i\hbar\sigma\tau/2}, \qquad (3.3.12)$$

and noting that $\exp(-i\tau\hat{p}/2)|q'\rangle = |q' - \hbar\tau/2\rangle$ etc., we find

$$W(p,q) = \frac{1}{(2\pi)^2} \int d\sigma \int d\tau \int dq' e^{i\sigma(q-q')} \langle q' + \hbar\tau/2|\rho|q' - \hbar\tau/2\rangle e^{i\tau p}. \qquad (3.3.13)$$

Finally, we carry out the σ-integration to obtain a delta function $\delta(q - q')$, which allows us to carry out the q'-integration, and introducing the notation $y = -\hbar\tau/2$, we write $W(p, q)$ in the usual form

$$W(p,q) = \frac{1}{\pi\hbar} \int dy e^{-i2yp/\hbar} \langle q - y|\rho|q + y\rangle. \qquad (3.3.14)$$

The Wigner function in the form (3.3.14) has been widely used in a host of problems; and we further elaborate on its connection with the P- and Q-distributions in the next section.

3.4 Generalized representation of the density operator and connection between the P-, Q-, and W-distributions

In the following, we present a generalized representation of the density operator originally due to Cohen and applied to quantum optics by Agarwal and Wolf. The P-, Q-, and W-representations can be derived as special cases of this generalized representation.

A generalized representation $F^{(\Omega)}(\alpha, \alpha^*)$ of the density operator is given by

$$\rho = \pi \int F^{(\Omega)}(\alpha, \alpha^*)\Delta^{(\Omega)}(\alpha - a, \alpha^* - a^\dagger)d^2\alpha, \tag{3.4.1}$$

where

$$\Delta^{(\Omega)}(\alpha - a, \alpha^* - a^\dagger) = \frac{1}{\pi^2}\int \exp[\Omega(\beta, \beta^*)] \times \exp[-\beta(\alpha^* - a^\dagger) + \beta^*(\alpha - a)]d^2\beta \tag{3.4.2}$$

Here $\Omega(\beta, \beta^*)$ (such that $\Omega(0,0) = 0$) is a function which characterizes different orderings. For example, when $\Omega(\beta, \beta^*) = -|\beta|^2/2$ we have $F^{(\Omega)}(\alpha, \alpha^*) \equiv P(\alpha, \alpha^*)$ and when $\Omega(\beta, \beta^*) = |\beta|^2/2$ we have $F^{(\Omega)}(\alpha, \alpha^*) \equiv Q(\alpha, \alpha^*)$.

To see these results explicitly, we first consider

$$\Omega(\beta, \beta^*) = -\frac{|\beta|^2}{2}. \tag{3.4.3}$$

It follows from Eqs. (2.2.6) and (2.2.7) that

$$\exp\left(-\frac{|\beta|^2}{2} + \beta a^\dagger - \beta^* a\right) = \exp(-\beta^* a)\exp(\beta a^\dagger), \tag{3.4.4}$$

and we obtain

$$\begin{aligned}\Delta^{(\Omega)}(\alpha - a, \alpha^* - a^\dagger) &= \frac{1}{\pi^2}\int e^{\beta^*(\alpha-a)}e^{-\beta(\alpha^*-a^\dagger)}d^2\beta \\ &= \frac{1}{\pi^3}\int\int e^{\beta^*(\alpha-a)}|\alpha_1\rangle\langle\alpha_1|e^{-\beta(\alpha^*-a^\dagger)}d^2\beta d^2\alpha_1 \\ &= \frac{1}{\pi^3}\int\int e^{\beta^*(\alpha-\alpha_1)-\beta(\alpha^*-\alpha_1^*)}|\alpha_1\rangle\langle\alpha_1|d^2\beta d^2\alpha_1 \\ &= \frac{1}{\pi}|\alpha\rangle\langle\alpha|.\end{aligned} \tag{3.4.5}$$

On substituting this expression for $\Delta^{(\Omega)}(\alpha - a, \alpha^* - a^\dagger)$ into Eq. (3.4.1) we recover the definition of the P-representation (Eq. (3.1.16)) with $F^{(\Omega)}(\alpha, \alpha^*) \equiv P(\alpha, \alpha^*)$.

On the other hand, if we choose $\Omega(\beta,\beta^*) = |\beta|^2/2$,

$$\Delta^{(\Omega)}(\alpha - a, \alpha^* - a^\dagger) = \frac{1}{\pi^2}\int e^{-\beta(\alpha^*-a^\dagger)}e^{\beta^*(\alpha-a)}d^2\beta. \tag{3.4.6}$$

It follows from Eq. (3.4.1) that

$$\frac{1}{\pi}\langle\alpha'|\rho|\alpha'\rangle = \int F^{(\Omega)}(\alpha,\alpha^*)\langle\alpha'|\Delta^{(\Omega)}(\alpha - a, \alpha^* - a^\dagger)|\alpha'\rangle d^2\alpha. \tag{3.4.7}$$

However, from Eq. (3.4.6),

$$\begin{aligned}\langle\alpha'|\Delta^{(\Omega)}(\alpha - a, \alpha^* - a^\dagger)|\alpha'\rangle &= \frac{1}{\pi^2}\int \langle\alpha'|e^{-\beta(\alpha^*-a^\dagger)}e^{\beta^*(\alpha-a)}|\alpha'\rangle d^2\beta \\ &= \frac{1}{\pi^2}\int e^{-\beta(\alpha^*-\alpha'^*)+\beta^*(\alpha-\alpha')}d^2\beta \\ &= \delta^{(2)}(\alpha-\alpha'). \end{aligned}\tag{3.4.8}$$

On carrying out the α-integration in Eq. (3.4.7) we recover Eq. (3.2.2) with $Q(\alpha,\alpha^*) \equiv F^{(\Omega)}(\alpha,\alpha^*)$.

Another distribution, the Wigner–Weyl distribution, is recovered for the proper choice of Ω, namely, $\Omega(\alpha,\alpha^*) = 0$. To that end, we invert Eq. (3.4.1) by using the function

$$\begin{aligned}\bar{\Delta}^{(\Omega)}(\alpha - a, \alpha^* - a^\dagger) = \frac{1}{\pi^2}\int &\exp[-\Omega(\beta,\beta^*)] \\ &\times\exp[\beta(\alpha^* - a^\dagger) - \beta^*(\alpha - a)]d^2\beta. \end{aligned}\tag{3.4.9}$$

Now, it can be shown that (see Problem 3.3)

$$\begin{aligned}&\mathrm{Tr}\left[\Delta^{(\Omega)}(\alpha - a, \alpha^* - a^\dagger)\bar{\Delta}^{(\Omega)}(\alpha' - a, \alpha'^* - a^\dagger)\right] \\ &= \frac{1}{\pi}\delta^{(2)}(\alpha-\alpha'). \end{aligned}\tag{3.4.10}$$

It then follows from Eq. (3.4.1) that

$$F^{(\Omega)}(\alpha,\alpha^*) = \mathrm{Tr}\left[\rho\bar{\Delta}^{(\Omega)}(\alpha - a, \alpha^* - a^\dagger)\right]. \tag{3.4.11}$$

From Eqs. (3.4.9) and (3.4.11), we obtain

$$\begin{aligned}&W(\alpha,\alpha^*) \\ &= \frac{1}{\pi^2}\int \mathrm{Tr}[\rho\exp(-\beta a^\dagger + \beta^* a)]\exp(\beta\alpha^* - \beta^*\alpha)d^2\beta, \end{aligned}\tag{3.4.12}$$

which, as expected, is the same as Eq. (3.3.6) with β replaced by $-i\beta$ and β^* by $i\beta^*$. Equations (3.1.14) for the P-representation and (3.2.2) for the Q-representation can be recovered from expression (3.4.11) for $\Omega(\beta,\beta^*) = -|\beta|^2/2$ and $\Omega(\beta,\beta^*) = |\beta|^2/2$, respectively.

In the following we derive an explicit expression for the Wigner–Weyl distribution $W(\alpha,\alpha^*)$. First we mention that $W(\alpha,\alpha^*)$ is the Fourier transform of the function $\mathrm{Tr}[\rho\exp(-\beta a^\dagger + \beta^* a)]/\pi^2$. We also note that $\exp(-2|\alpha|^2)$ is the Fourier transform of $\exp(-|\beta|^2/2)/2\pi$, i.e.,

$$\exp(-2|\alpha|^2) = \frac{1}{2\pi}\int \exp\left(-\frac{1}{2}|\beta|^2\right)\exp(\beta\alpha^* - \beta^*\alpha)d^2\beta. \quad (3.4.13)$$

It then follows from the convolution theorem that

$$W(\alpha,\alpha^*)\exp(-2|\alpha|^2) = \int C(\beta,\beta^*)\exp(\beta\alpha^* - \beta^*\alpha)d^2\beta, \quad (3.4.14)$$

where $C(\beta,\beta^*)$ is the convolution product

$$C(\beta,\beta^*) = \frac{1}{2\pi^3}\int \mathrm{Tr}\{\rho\exp[-(\beta-\beta_1)a^\dagger + (\beta^* - \beta_1^*)a]\}$$
$$\exp\left(-\frac{1}{2}|\beta_1|^2\right)d^2\beta_1. \quad (3.4.15)$$

An explicit expression for $C(\beta,\beta^*)$ can be obtained by using the identity (2.2.7) and inserting the resolution of the identity in terms of coherent states (Eq. (2.4.6)) as follows:

$$\begin{aligned}C(\beta,\beta^*) &= \frac{1}{2\pi^5}\int\int\int \mathrm{Tr}\{\rho|\beta_2\rangle\langle\beta_2|\exp[-(\beta-\beta_1)a^\dagger]\\ &\quad\times\exp[(\beta^*-\beta_1^*)a]|\beta_3\rangle\langle\beta_3|\}\\ &\quad\times\exp\left(-\frac{1}{2}|\beta-\beta_1|^2 - \frac{1}{2}|\beta_1|^2\right)d^2\beta_1 d^2\beta_2 d^2\beta_3\\ &= \frac{1}{2\pi^5}\int\int\int\langle\beta_3|\rho|\beta_2\rangle\langle\beta_2|\beta_3\rangle\\ &\quad\times\exp\left[-(\beta-\beta_1)\beta_2^* + (\beta^*-\beta_1^*)\beta_3\right.\\ &\qquad\left.-\frac{1}{2}|\beta-\beta_1|^2 - \frac{1}{2}|\beta_1|^2\right]\\ &\quad\times d^2\beta_1 d^2\beta_2 d^2\beta_3. \end{aligned} \quad (3.4.16)$$

On carrying out the integrations over β_1, Eq. (3.4.16) reduces to

$$\begin{aligned}C(\beta,\beta^*) &= \frac{1}{2\pi^4}\int\int\langle\beta_3|\rho|\beta_2\rangle\langle\beta/2|\beta_3\rangle\langle\beta_2|-\beta/2\rangle d^2\beta_2 d^2\beta_3\\ &= \frac{1}{2\pi^2}\langle\beta/2|\rho|-\beta/2\rangle.\end{aligned} \quad (3.4.17)$$

Finally, on substituting for $C(\beta,\beta^*)$ from Eq. (3.4.17) into Eq. (3.4.14) and changing the variables of integration from β,β^* to $-2\beta,-2\beta^*$, we obtain

$$\begin{aligned}&W(\alpha,\alpha^*)\\ &= \frac{2}{\pi^2}\exp(2|\alpha|^2)\int\langle-\beta|\rho|\beta\rangle\exp[-2(\beta\alpha^* - \beta^*\alpha)]d^2\beta.\end{aligned} \quad (3.4.18)$$

This expression, which is very similar to the corresponding expression for P-representation (Eq. (3.1.19)), can be used to evaluate the Wigner–Weyl distribution for the given density operator of the field.

3.5 Q-representation for a squeezed coherent state

In this section, we derive the Q-representation for the squeezed coherent state $|\beta, \xi\rangle$. According to Eq. (3.2.2)

$$Q(\alpha, \alpha^*) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle = \frac{1}{\pi}|\langle\alpha|\beta, \xi\rangle|^2. \tag{3.5.1}$$

Now

$$\langle\alpha|\beta, \xi\rangle = \langle\alpha|S(\xi)D(\beta)|0\rangle = \langle\alpha|S(\xi)|\beta\rangle. \tag{3.5.2}$$

We therefore need to calculate the function $\langle\alpha|S(\xi)|\beta\rangle$.

It follows, on using the properties of the coherent state and the transformation property (2.7.7) of $S(\xi)$ that

$$\begin{aligned}\langle\alpha|S(\xi)|\beta\rangle &= \frac{1}{\alpha^*}\langle\alpha|a^\dagger S(\xi)|\beta\rangle \\ &= \frac{1}{\alpha^*}\langle\alpha|S(\xi)S^\dagger(\xi)a^\dagger S(\xi)|\beta\rangle \\ &= \frac{1}{\alpha^*}\langle\alpha|S(\xi)(a^\dagger \cosh r - ae^{-i\theta}\sinh r)|\beta\rangle \\ &= \frac{1}{\alpha^*}\left[\cosh r\left(\frac{\partial}{\partial\beta} + \frac{1}{2}\beta^*\right) - e^{-i\theta}\beta\sinh r\right]\langle\alpha|S(\xi)|\beta\rangle.\end{aligned} \tag{3.5.3}$$

The function $\langle\alpha|S(\xi)|\beta\rangle$ therefore satisfies the following differential equation

$$\left[\cosh r\frac{\partial}{\partial\beta} - \beta e^{-i\theta}\sinh r + \left(\frac{1}{2}\beta^*\cosh r - \alpha^*\right)\right]\langle\alpha|S(\xi)|\beta\rangle = 0. \tag{3.5.4}$$

The solution of this equation is

$$\begin{aligned}&\langle\alpha|S(\xi)|\beta\rangle \\ &= K\exp\left(-\frac{1}{2}|\beta|^2 + \alpha^*\beta\operatorname{sech} r + \frac{1}{2}e^{-i\theta}\beta^2\tanh r\right).\end{aligned} \tag{3.5.5}$$

The form of K, which may depend upon $\alpha, \alpha^*, \beta^*, r$, and θ, can be determined using the unitarity of $S(\xi)$. It follows from

$$\langle\alpha|S(\xi)|\beta\rangle^* = \langle\beta|S^\dagger(\xi)|\alpha\rangle = \langle\beta|S(-\xi)|\alpha\rangle, \tag{3.5.6}$$

that

$$K^*(\alpha,\alpha^*,\beta^*,r,\theta)\exp\left[-\frac{1}{2}|\beta|^2+\frac{1}{2}e^{i\theta}(\beta^*)^2\tanh r\right]$$
$$=K(\beta,\beta^*,\alpha^*,r,\theta+\pi)\exp\left(-\frac{1}{2}|\alpha|^2-\frac{1}{2}e^{-i\theta}\alpha^2\tanh r\right). \quad (3.5.7)$$

The form of K is therefore

$$K(\alpha,\alpha^*,\beta^*,r,\theta)$$
$$=(\text{sech }r)^{1/2}\exp\left[-\frac{1}{2}|\alpha|^2-\frac{1}{2}e^{i\theta}(\alpha^*)^2\tanh r\right]. \quad (3.5.8)$$

The coefficient $(\text{sech }r)^{1/2}$ is chosen so that the normalization condition

$$\frac{1}{\pi}\int|\langle\alpha|S(\xi)|\beta\rangle|^2d^2\alpha=1 \quad (3.5.9)$$

is satisfied.

On substituting this expression for K in Eq. (3.5.5) we obtain

$$\langle\alpha|S(\xi)|\beta\rangle=(\text{sech }r)^{1/2}\exp\left\{-\frac{1}{2}(|\alpha|^2+|\beta|^2)+\alpha^*\beta\,\text{sech }r\right.$$
$$\left.-\frac{1}{2}\left[e^{i\theta}(\alpha^*)^2-e^{-i\theta}\beta^2\right]\tanh r\right\}. \quad (3.5.10)$$

The Q-representation for the state $|\beta,\xi\rangle$ is therefore

$$Q(\alpha,\alpha^*)=\frac{\text{sech }r}{\pi}\exp\left\{-(|\alpha|^2+|\beta|^2)+(\alpha^*\beta+\beta^*\alpha)\text{sech }r\right.$$
$$\left.-\frac{1}{2}[e^{i\theta}(\alpha^{*2}-\beta^{*2})+e^{-i\theta}(\alpha^2-\beta^2)]\tanh r\right\}. \quad (3.5.11)$$

In Fig. 3.2, $Q(\alpha,\alpha^*)\equiv Q(X_1,X_2)$ ($X_1=(\alpha+\alpha^*)/2$, $X_2=(\alpha-\alpha^*)/2i$) is plotted as a function of the amplitudes X_1,X_2 of the two quadratures. We clearly see the unequal variances in X_1 and X_2 in the state $|\alpha,\xi\rangle$. We can employ expression (3.5.10) for $\langle\alpha|S(\xi)|\beta\rangle$ to calculate the photon distribution function of a squeezed coherent state.

The photon distribution function $p(n)$ for the field in state $|\beta,\xi\rangle$ is given by

$$p(n)=|\langle n|\beta,\xi\rangle|^2. \quad (3.5.12)$$

The quantity $\langle n|\beta,\xi\rangle$ can be determined by writing

$$\langle\alpha|\beta,\xi\rangle=\sum_{n=0}^{\infty}\langle\alpha|n\rangle\langle n|\beta,\xi\rangle=e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{(\alpha^*)^n}{\sqrt{n!}}\langle n|\beta,\xi\rangle, \quad (3.5.13)$$

and expanding the right-hand side of Eq. (3.5.10) in powers of α^* by means of the generating function for the Hermite polynomials $H_n(z)$:

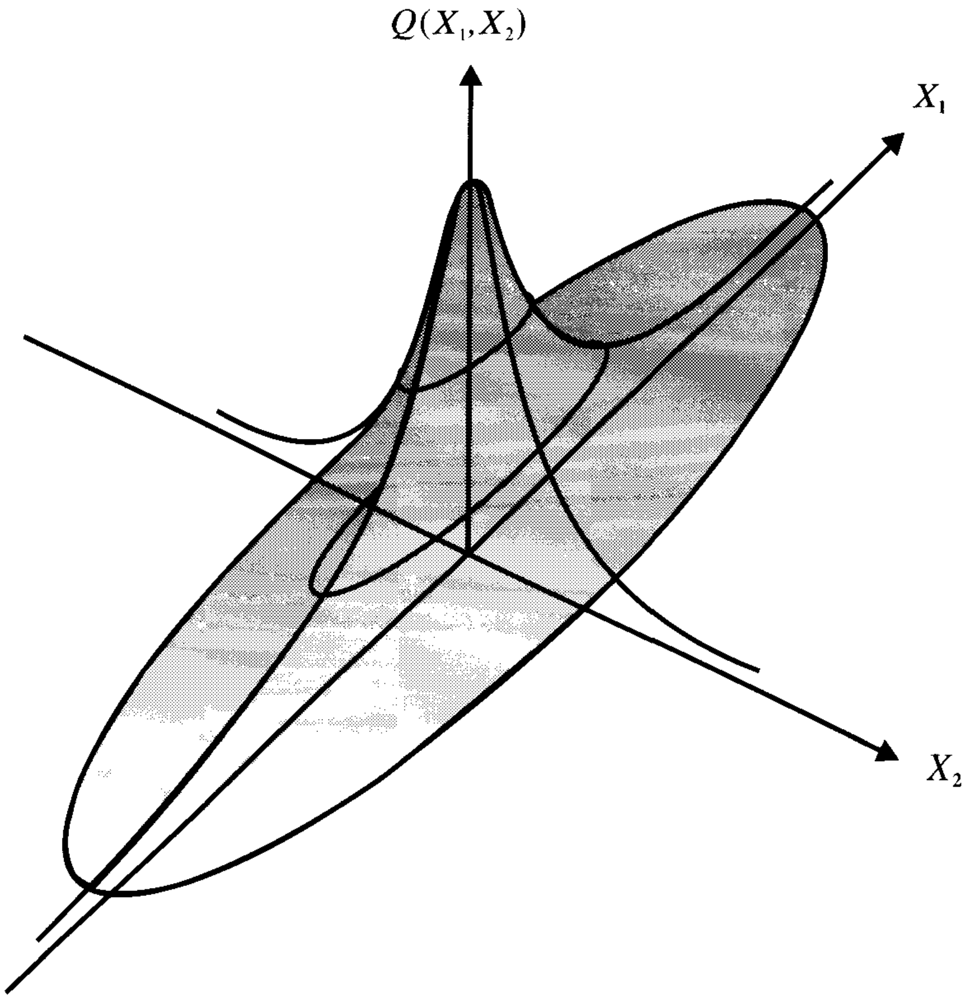


Fig. 3.2
A plot of $Q(\alpha,\alpha^*) \equiv Q(X_1,X_2)$ as a function of the amplitudes X_1 and X_2 in a squeezed coherent state. (From H. P. Yuen, *Phys. Rev. A* **13**, 2226 (1976).)

$$\exp(2zt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(z)t^n}{n!}. \tag{3.5.14}$$

On comparing the resulting expansion with the expansion in Eq. (3.5.13), it follows that

$$\langle n|\beta,\xi\rangle = \frac{(e^{i\theta}\tanh r)^{n/2}}{2^{n/2}(n!\cosh r)^{1/2}} \exp\left[-\frac{1}{2}(|\beta|^2 - e^{-i\theta}\beta^2 \tanh r)\right] \times H_n\left(\frac{\beta e^{-i\theta/2}}{\sqrt{2\cosh r \sinh r}}\right). \tag{3.5.15}$$

The photon distribution function $p(n)$ for an ideal squeezed state is therefore given by

$$p(n) = \frac{(\tanh r)^n}{2^n n! \cosh r} \exp\left\{-|\beta|^2 + \frac{1}{2}[e^{-i\theta}\beta^2 + e^{i\theta}(\beta^*)^2]\tanh r\right\} \times \left|H_n\left(\frac{\beta e^{-i\theta/2}}{\sqrt{2\cosh r \sinh r}}\right)\right|^2. \tag{3.5.16}$$

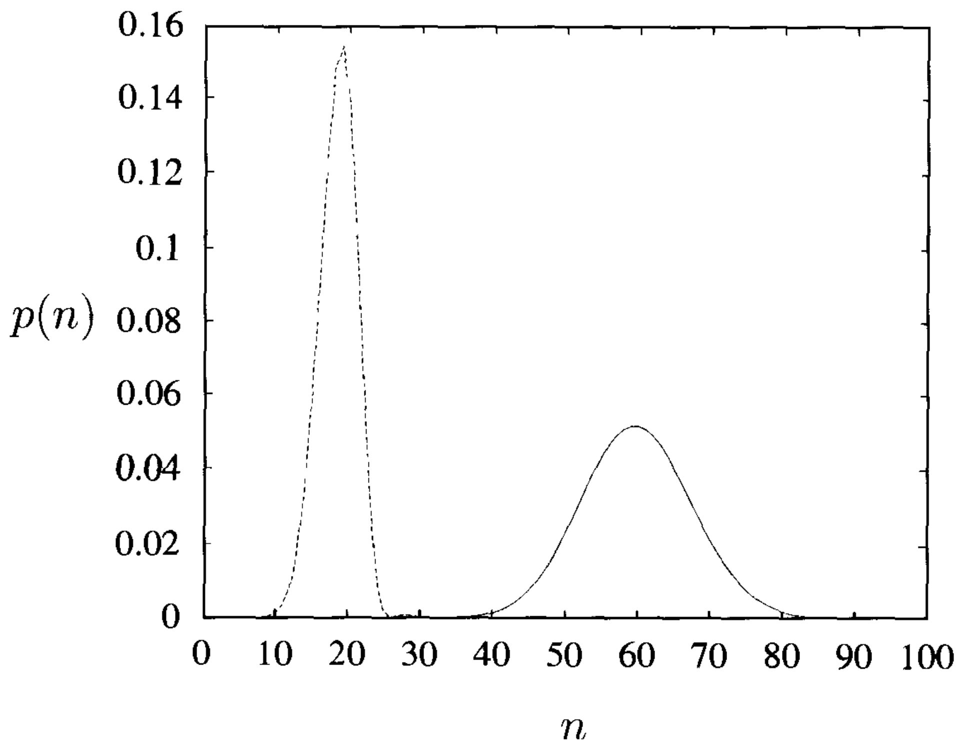


Fig. 3.3 Comparison of photon distribution function for a coherent state $|\alpha\rangle$ with $|\alpha|^2 = 60$ (solid line) with the squeezed coherent state $|\beta, \xi\rangle$ ($\beta = |\beta| \exp(i\phi)$, $\xi = r \exp(i\theta)$) with $|\beta|^2 = 60$, $r = 0.6$, and $\phi = \theta/2$ (dashed line).

Generally, sources of squeezing produce a radiation field in a squeezed vacuum state $|0, \xi\rangle$. The detection schemes, however, add a coherent component to it. The detected state is therefore described by the distribution (3.5.16). The fluctuations in the mean number of photons can be found either from Eq. (3.5.16), by using

$$\langle n^r \rangle = \sum_{n=0}^{\infty} n^r p(n), \tag{3.5.17}$$

or through the use of the unitary transformation properties of the squeeze operator (2.7.6) and (2.7.7). We obtain

$$(\Delta n)^2 = |\beta|^2 [\cosh 4r - \cos(\theta - 2\phi) \sinh 4r] + 2 \sinh^2 r \cosh^2 r. \tag{3.5.18}$$

In the following, we discuss three cases of interest. First, when $|\beta|^2 \gg \sinh^2 r$, the coherent component is larger than the squeeze component. Figure 3.3 compares the probability distribution for a squeezed state with a coherent state. If the squeezing is along the coherent amplitude, the state has sub-Poissonian photon statistics. In the second case (Fig. 3.4) when the squeeze component is larger than the coherent component and squeezing is along the coherent amplitude, the squeezed state exhibits oscillations. The main peak as well as the subsequent peaks are narrower than the corresponding $\sqrt{n}$ value. But the overall distribution shows super-Poissonian statistics.

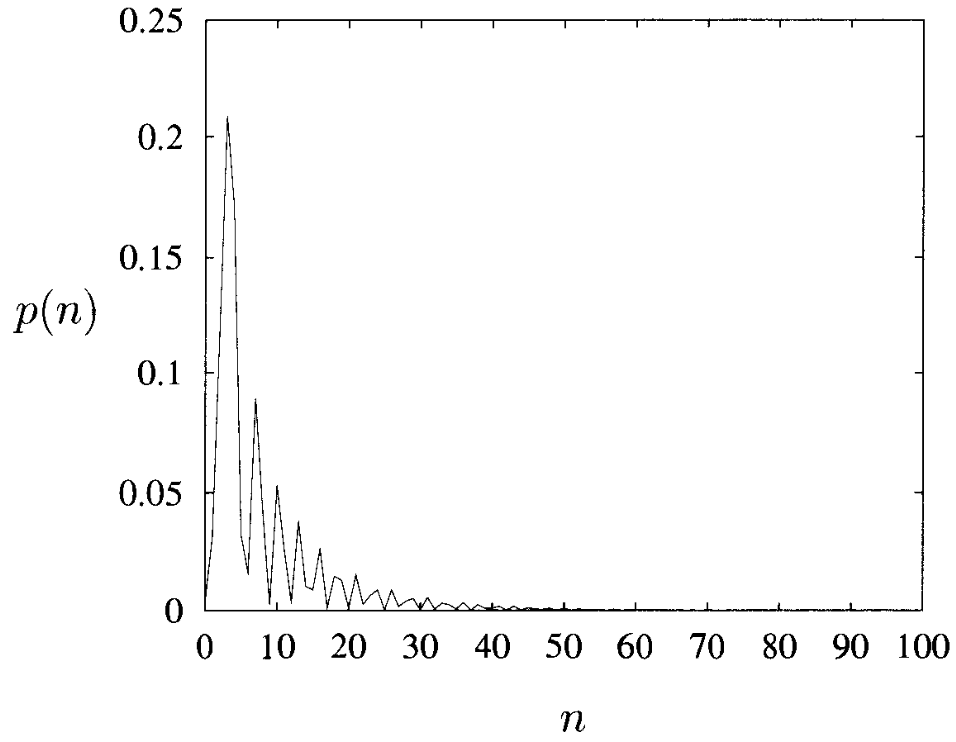


Fig. 3.4
Photon distribution function $p(n)$ for a two-photon coherent state (Eq.(3.5.16)) for $|\beta|^2 = 60$, $r = 1.6$. The squeeze component is larger than the coherent component and squeezing is along the coherent amplitude.

Finally, for zero displacement, i.e., for the squeezed vacuum state, the distribution function (3.5.16) reduces to

$$\begin{aligned} p(2n) &= (\cosh r)^{-1} \frac{(2n)!}{(n!)^2} \left(\frac{1}{2} \tanh r \right)^{2n}, \\ p(2n+1) &= 0. \end{aligned} \tag{3.5.19}$$

In the above equations, a nonzero value for even terms arises due to squeezing of the vacuum and clearly shows the 'two-photon' nature of the field. Figure 3.5 shows a plot of the probability distribution (3.5.19). The distribution peaks sharply at $n = 0$ and has a very long tail similar to a thermal distribution.

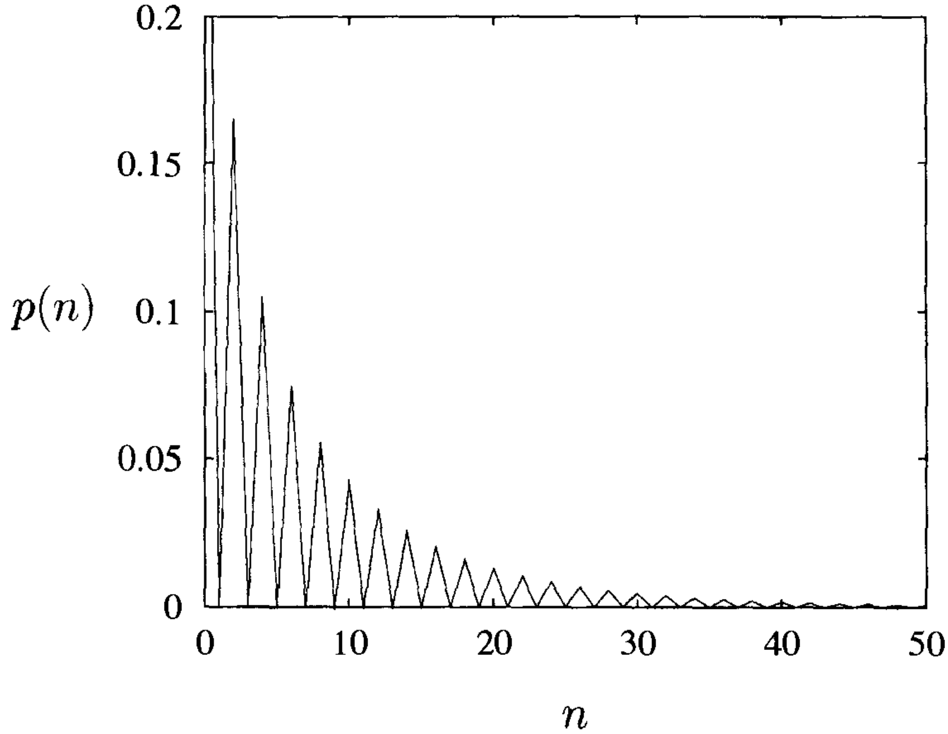
3.A Verifying equations (3.1.12a, 3.1.12b)

It can be verified that the two-dimensional delta function has the form (3.1.12a)

$$\delta(\alpha^* - a^\dagger)\delta(\alpha - a) = \frac{1}{\pi^2} \int \exp[-\beta(\alpha^* - a^\dagger)] \exp[\beta^*(\alpha - a)] d^2\beta \tag{3.A.1}$$

by taking the expectation values in a coherent state $|\gamma\rangle$ of both sides of Eq. (3.A.1). Indeed, on doing so and utilizing the fact that $|\gamma\rangle$ is an

Fig. 3.5
Photon distribution function $p(n)$ for a squeezed vacuum state for $r = 1.6$.



eigenstate of the annihilation operator a with an eigenvalue γ, we get

$$\delta(\alpha^* - \gamma^*)\delta(\alpha - \gamma) = \frac{1}{\pi^2}\int \exp[-\beta(\alpha^* - \gamma^*)]\exp[\beta^*(\alpha - \gamma)]d^2\beta. \tag{3.A.2}$$

If we write $\alpha = x_\alpha + iy_\alpha$, $\beta = x_\beta + iy_\beta$, and $\gamma = x_\gamma + iy_\gamma$, then $d^2\beta = dx_\beta dy_\beta$ and the right-hand side of Eq. (3.A.2) becomes

$$\begin{aligned}
&\frac{1}{\pi^2}\int \exp[-\beta(\alpha^* - \gamma^*)]\exp[\beta^*(\alpha - \gamma)]d^2\beta \\
&= \frac{1}{\pi^2}\int \exp\left\{2i[x_\beta(y_\alpha - y_\gamma) - y_\beta(x_\alpha - x_\gamma)]\right\}dx_\beta dy_\beta \\
&= \left(\frac{1}{2\pi}\right)^2 \int\int \exp\left\{i[x_\beta(y_\alpha - y_\gamma) - y_\beta(x_\alpha - x_\gamma)]\right\}dx_\beta dy_\beta \\
&= \delta[\mathrm{Im}(\alpha - \gamma)]\delta[\mathrm{Re}(\alpha - \gamma)] \\
&\equiv \delta(\alpha - \gamma)\delta(\alpha^* - \gamma^*),
\end{aligned} \tag{3.A.3}$$

where we have replaced $2x_\beta$ and $2y_\beta$ by x_β and y_β, respectively, in the second line and used the following expression for the delta function

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk. \tag{3.A.4}$$

Equation (3.1.12b)

$$\delta(\alpha^* - a^\dagger)\delta(\alpha - a)$$
$$= \frac{1}{\pi^2}\int \exp[-i\beta(\alpha^* - a^\dagger)]\exp[-i\beta^*(\alpha - a)]d^2\beta \tag{3.A.5}$$

can be obtained from (3.A.1) simply by changing the variables $\beta \to i\beta$ and $\beta^* \to -i\beta^*$.

Another formula for the antinormally ordered two-dimensional delta function, namely,

$$\delta(\alpha - a)\delta(\alpha^* - a^\dagger)$$
$$= \frac{1}{\pi^2}\int \exp[\beta^*(\alpha - a)]\exp[-\beta(\alpha^* - a^\dagger)]d^2\beta, \tag{3.A.6}$$

which has been used to define the Q-representation (Eq. (3.2.1)), can be proven by inserting

$$\frac{1}{\pi}\int |\gamma\rangle\langle\gamma| d^2\gamma = 1, \tag{3.A.7}$$

as follows:

$$\begin{aligned}
&\frac{1}{\pi^2}\int \exp[\beta^*(\alpha - a)]\exp[-\beta(\alpha^* - a^\dagger)]d^2\beta \\
&= \frac{1}{\pi^3}\int\int e^{\beta^*(\alpha-a)}|\gamma\rangle\langle\gamma| e^{-\beta(\alpha^*-a^\dagger)}d^2\beta d^2\gamma \\
&= \frac{1}{\pi^3}\int\int e^{\beta^*(\alpha-\gamma)}|\gamma\rangle\langle\gamma| e^{-\beta(\alpha^*-\gamma^*)}d^2\beta d^2\gamma \\
&= \frac{1}{\pi}\int \delta(\alpha-\gamma)|\gamma\rangle\langle\gamma|\delta(\alpha^*-\gamma^*)d^2\gamma \\
&= \delta(\alpha - a)\left(\frac{1}{\pi}\int |\gamma\rangle\langle\gamma| d^2\gamma\right)\delta(\alpha^* - a^\dagger) \\
&= \delta(\alpha - a)\delta(\alpha^* - a^\dagger).
\end{aligned} \tag{3.A.8}$$

3.B c-number function correspondence for the Wigner–Weyl distribution

Given an operator $O(a, a^\dagger)$ and the Wigner–Weyl distribution $W(\alpha, \alpha^*)$, we calculate the c-number function $O_S(\alpha, \alpha^*)$ such that

$$\langle O(a, a^\dagger)\rangle = \mathrm{Tr}(O\rho) = \int d^2\alpha O_S(\alpha, \alpha^*)W(\alpha, \alpha^*). \tag{3.B.1}$$

Recall that the Wigner–Weyl distribution is defined as (Eqs. (3.3.6) and (3.3.7))

$$W(\alpha,\alpha^*) = \frac{1}{\pi^2}\int d^2\beta \mathrm{Tr}\left(e^{i\beta^* a + i\beta a^\dagger}\rho\right)e^{-i\beta^*\alpha - i\beta\alpha^*} \tag{3.B.2}$$

with the characteristic function

$$\begin{aligned} C^{(s)}(\beta,\beta^*) &= \int e^{i\beta^*\alpha+i\beta\alpha^*}W(\alpha,\alpha^*)d^2\alpha \\ &= \mathrm{Tr}\left(e^{i\beta^* a+i\beta a^\dagger}\rho\right) \\ &= \mathrm{Tr}\left(e^{i\beta a^\dagger}e^{i\beta^* a}e^{-|\beta|^2/2}\rho\right). \end{aligned} \tag{3.B.3}$$

where, in the last line, we use the Baker–Hausdorff formula (2.2.7). Now for any normally ordered operator $O(a,a^\dagger)$, one can write

$$O(a,a^\dagger) = \sum_{n,m} c_{n,m}a^{\dagger n}a^m. \tag{3.B.4}$$

It can be easily found that

$$\langle a^\dagger\rangle = \left[\frac{\partial}{\partial(i\beta)} + \frac{\beta^*}{2i}\right] C^{(s)}(\beta,\beta^*)\Big|_{\beta^*=\beta=0},$$

and

$$\langle a\rangle = \left[\frac{\partial}{\partial(i\beta^*)} + \frac{\beta}{2i}\right] C^{(s)}(\beta,\beta^*)\Big|_{\beta^*=\beta=0}.$$

Then we have

$$\begin{aligned} &\langle O(a,a^\dagger)\rangle \\ &= \sum_{n,m} c_{n,m}\left[\frac{\partial}{\partial(i\beta)} + \frac{\beta^*}{2i}\right]^n \left[\frac{\partial}{\partial(i\beta^*)} + \frac{\beta}{2i}\right]^m C^{(s)}(\beta,\beta^*)\Big|_{\beta^*=\beta=0} \\ &= \int d^2\alpha \sum_{n,m} c_{n,m}\left[\frac{\partial}{\partial(i\beta)} + \frac{\beta^*}{2i}\right]^n \\ &\qquad \times \left[\frac{\partial}{\partial(i\beta^*)} + \frac{\beta}{2i}\right]^m e^{i\beta^*\alpha+i\beta\alpha^*}\Big|_{\beta^*=\beta=0} W(\alpha,\alpha^*) \\ &\equiv \int d^2\alpha O_S(\alpha,\alpha^*)W(\alpha,\alpha^*), \end{aligned} \tag{3.B.5}$$

which yields

$$\begin{aligned} &O_S(\alpha,\alpha^*) \\ &= \sum_{n,m} c_{n,m}\left[\frac{\partial}{\partial(i\beta)} + \frac{\beta^*}{2i}\right]^n \left[\frac{\partial}{\partial(i\beta^*)} + \frac{\beta}{2i}\right]^m e^{i\beta^*\alpha+i\beta\alpha^*}\Big|_{\beta^*=\beta=0}. \end{aligned} \tag{3.B.6}$$

Equation (3.B.6) is our desired result. Consider some examples:

(a) $O(a, a^\dagger) = a^\dagger a$

$$
\begin{aligned}
&O_S(\alpha, \alpha^*) \\
&= \left[\frac{\partial}{\partial(i\beta)} + \frac{\beta^*}{2i}\right]\left[\frac{\partial}{\partial(i\beta^*)} + \frac{\beta}{2i}\right] e^{i\beta^*\alpha + i\beta\alpha^*}\Big|_{\beta^*=\beta=0} \\
&= \left[\frac{\partial}{\partial(i\beta)} + \frac{\beta^*}{2i}\right]\left(\alpha + \frac{\beta}{2i}\right) e^{i\beta^*\alpha + i\beta\alpha^*}\Big|_{\beta^*=\beta=0} \\
&= \alpha^*\alpha - \frac{1}{2}.
\end{aligned} \tag{3.B.7}
$$

(b) $O(a, a^\dagger) = a^{\dagger 2} a$

$$
\begin{aligned}
&O_S(\alpha, \alpha^*) \\
&= \left(\frac{\partial}{\partial(i\beta)} + \frac{\beta^*}{2i}\right)^2\left(\frac{\partial}{\partial(i\beta^*)} + \frac{\beta}{2i}\right) e^{i\beta^*\alpha + i\beta\alpha^*}\Big|_{\beta^*=\beta=0} \\
&= \left(\frac{\partial}{\partial(i\beta)} + \frac{\beta^*}{2i}\right)^2\left(\alpha + \frac{\beta}{2i}\right) e^{i\beta^*\alpha + i\beta\alpha^*}\Big|_{\beta^*=\beta=0} \\
&= \left(\frac{\partial}{\partial(i\beta)} + \frac{\beta^*}{2i}\right) \\
&\quad \left[\left(\alpha^* + \frac{\beta^*}{2i}\right)\left(\alpha + \frac{\beta}{2i}\right) - \frac{1}{2}\right] e^{i\beta^*\alpha + i\beta\alpha^*}\Big|_{\beta^*=\beta=0} \\
&= \left\{-\frac{1}{2}\left(\alpha^* + \frac{\beta^*}{2i}\right) + \left(\alpha^* + \frac{\beta^*}{2i}\right)\right. \\
&\quad \left.\left[\left(\alpha^* + \frac{\beta^*}{2i}\right)\left(\alpha + \frac{\beta}{2i}\right) - \frac{1}{2}\right]\right\} e^{i\beta^*\alpha + i\beta\alpha^*}\Big|_{\beta^*=\beta=0} \\
&= -\frac{\alpha^*}{2} + \alpha^*\left(\alpha^*\alpha - \frac{1}{2}\right) \\
&= \alpha^{*2}\alpha - \alpha^*.
\end{aligned} \tag{3.B.8}
$$

The operator corresponding to the Wigner distribution function in the coordinate-momentum representation is given by Cohen (1986).

Problems

3.1 Show that

$$\frac{1}{2}\langle a a^\dagger + a^\dagger a\rangle = \int W(\alpha, \alpha^*)|\alpha|^2 d^2\alpha,$$

where $W(\alpha, \alpha^*)$ is the Wigner–Weyl distribution.

3.2 Show that

$$\mathrm{Tr}[D(\alpha)] = \pi\delta^{(2)}(\alpha),$$
$$\mathrm{Tr}[D(\alpha)D^\dagger(\alpha')] = \pi\delta^{(2)}(\alpha - \alpha'),$$

where $D(\alpha)$ is the displacement operator. Using these results, show that

$$\mathrm{Tr}[\Delta^{(\Omega)}(\alpha - a, \alpha^* - a^\dagger)\bar{\Delta}^{(\Omega)}(\alpha' - a, \alpha^{*\prime} - a^\dagger)]$$
$$= \frac{1}{\pi}\delta^{(2)}(\alpha - \alpha'),$$

The operators $\Delta^{(\Omega)}$ and $\bar{\Delta}^{(\Omega)}$ are defined in Eqs. (3.4.2) and (3.4.9), respectively.

3.3 Show that the Wigner–Weyl distribution $W(\alpha, \alpha^*)$ can be expressed in terms of the P-representation $P(\alpha, \alpha^*)$ via the relation

$$W(\alpha, \alpha^*) = \frac{2}{\pi}\int P(\beta, \beta^*)\exp(-2|\alpha - \beta|^2)d^2\beta.$$

3.4 Determine $Q(\alpha, \alpha^*)$ and $W(\alpha, \alpha^*)$ for a coherent state and a thermal state.